

The Fitch Trick: Solutions & Variations

ଫିଚ୍ ଡାସ ଖେଳ ଏବଂ ଏହାର ରହସ୍ୟ

ନୀଳଧାରା ମିଶ୍ର • Neeldhara Misra

ସାରାଂଶ. In last issue's puzzle, five cards from the audience went to Badarika, who tucked one away and laid the remaining four face-up in a row; Aalaya, absent throughout, returned and named the hidden card. How? This article delivers the complete answer. We give the full strategy for a standard 52-card deck, then ask how far the trick can be pushed: it turns out 124 is the largest deck size for which a winning strategy can exist, and, perhaps more surprisingly, one always does. We prove this bound, construct an explicit scheme that realizes it, and offer a gentler warm-up that reaches 124 by bending one small rule. The article closes with a tour of variations, including a version where Aalaya predicts the outcome of a coin flip and one where the four cards may be placed face-up or face-down.

In the trick we presented in the last issue, Badarika the assistant studied five cards an audience handed her, slipped one away, and laid the other four face-up in a row. The magician Aalaya—who had been out of the room throughout—walked in, glanced at the four cards, and named the fifth.

Here we revisit this magic trick and discuss how it works: we begin with the standard deck and then show how to perform the same effect with a massively extended deck of size 124, and conclude with a discussion of several variations.

The Two Levers

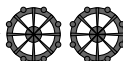
Aalaya seems to face long odds. Badarika pockets one of five cards, lays the other four face-up in a row, and Aalaya—standing across the room, seeing only those four—must name the hidden card from the remaining 48 possibilities. But look more carefully at what

Badarika controls. She chooses which of the five cards to hide, picking from five candidates; and she chooses the order in which the remaining four cards are placed, one of $4! = 24$ arrangements. These are two separate levers, and the trick turns on using both together.

The first lever comes from a simple counting observation. Five cards drawn from a 52-card deck must include at least two of the same suit, because there are only four suits and five cards—this is the so-called the pigeonhole principle at work. Indeed, if no suit repeated, we could have at most four cards, which contradicts the situation that there are, in fact, five. Note that there may be more than one pair of cards of the same suit and there may also be more than two cards that belong to the same suit. The claim is only that at *any* subset of five cards from a standard deck of 52 cards *is guaranteed* to contain a pair of cards that have the same suit.

Badarika finds that matching pair and hides one of them. The other she lays face-up in the first position. The moment Aalaya sees that first card, she knows the hidden card's suit. A problem with 48 unknowns suddenly becomes a problem with at most 12: the hidden card is one of the twelve remaining cards of that suit.

The second lever handles the rank. Picture the thirteen ranks—ace, two, three, up to ten, jack, queen, king—arranged around a circle, the ace at the top and each rank one step clockwise from the last (see Figure 1). Any two positions on this clock are at most six steps apart the short way round. To see this, say the numbers are x and y . Try going clockwise from x to y . If you consume at most six steps, you are done, if not, then you



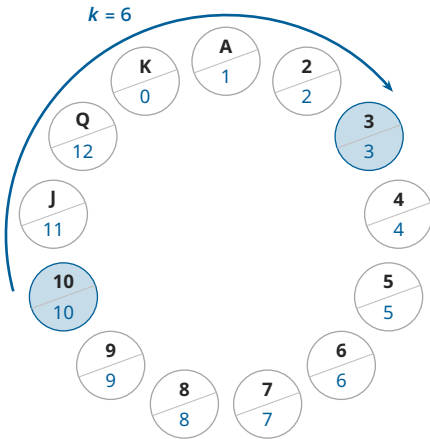


Figure 1: The thirteen ranks on a clock, each split to show its remainder mod 13 underneath (so *K* is 0, *Q* is 12, and so on). In this example, the cards we have have ranks 3 and 10. Note that 10 is seven steps ahead of 3, and 3 lies six steps past the 10: so we reveal the 10, hide the 3, and $k = 6$. In terms of remainders, $10 + 6 = 16 \equiv 3 \pmod{13}$.

consumed at least seven, so going counter clockwise from x to y (or equivalently, going clockwise from y to x) will consume at most six steps, since there are only thirteen steps available altogether.

So among Badarika's two same-suit cards, one always sits at most six clockwise steps ahead of the other. She reveals the card she counts *from* and keeps the one that is k steps *ahead* hidden, where k lies between 1 and 6. By choosing which of the pair to hide, she guarantees the hidden card is always ahead and never more than six steps away.

Now the whole problem collapses to a single number: Aalaya needs to receive k , somewhere from 1 to 6. The three remaining cards are exactly the tool. Beforehand the partners fix an ordering of the entire deck—by rank first, and among equal ranks by suit in the order clubs, hearts, spades, diamonds¹. Under this ordering any three distinct cards

fall into one Lowest, one Middle, one Highest. There are $3! = 6$ ways to arrange three distinct things, and the partners agree on the table below.

LMH	LHM	MLH	MHL	HLM	HML
1	2	3	4	5	6

Putting it together, here is Badarika's recipe. Find the two same-suit cards. Decide which to hide—the one that is 1 to 6 clockwise steps ahead of the other—and lay the other face-up first. Then arrange the remaining three in the order that encodes k . Aalaya decodes in reverse: read the suit from the first card; read k from the Low-Middle-High pattern of the other three; count k steps clockwise from the first card's rank; and name that card in the suit she already knows.

Let's go over a concrete example. Suppose the five cards are the $4\spadesuit$, $5\heartsuit$, $2\clubsuit$, $7\clubsuit$, $6\diamondsuit$ —exactly the hand posed as a problem in the companion article. The two clubs are the matching pair. Going clockwise from the 2 to the 7 is exactly five steps, so $k = 5$: Badarika hides the $7\clubsuit$ and sets down the $2\clubsuit$ first. The three left over are $4\spadesuit$, $5\heartsuit$, $6\diamondsuit$; in the global order their ranks 4, 5, 6 are distinct, so $L = 4\spadesuit$, $M = 5\heartsuit$, $H = 6\diamondsuit$. To send $k = 5$ she needs HLM, so she lays the $6\diamondsuit$, then the $4\spadesuit$, then the $5\heartsuit$. Aalaya sees the row in Figure 2. She reads clubs from the first card, recognises HLM = 5 in the rest, counts five steps from rank 2 to rank 7, and names the $7\clubsuit$. In the notation of the companion piece, the hand $a = \langle 4\spadesuit, 5\heartsuit, 2\clubsuit, 7\clubsuit, 6\diamondsuit \rangle$ maps to $\beta = \langle 2\clubsuit, 6\diamondsuit, 4\spadesuit, 5\heartsuit \rangle$.

Rehearsing until this is automatic matters more than performers like to admit. The good news is that the mechanics do become second nature with practice. However, if you are going to repeat the trick though, there is one structural tell: the suit-revealing card always sits first, and an alert audience that

¹This is sometimes called the CHaSeD order, which makes it easy to remember. Of course, you can pick

any order you like, as long as you and your partner both swear by the same one!

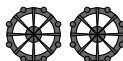




Figure 2: The four cards Aalaya sees, in order, and the card she must name. First card $\Rightarrow$ suit (clubs); the rest spell $HLM = 5$; $2 + 5 = 7$, so the hidden card is the $7 \clubsuit$.

sees several rounds will start to wonder why position one seems special. There is a way to workaround this, and it is worth keeping in mind—we will return to it among the variations.

Problem 1

Warm up as the magician. Using the basic scheme above (deck order $A, 2, \dots, K$, suits $\clubsuit \diamond \heartsuit \spadesuit$), name the hidden card for each row:

- (a) $\langle 2 \clubsuit, 6 \spadesuit, 6 \heartsuit, K \clubsuit \rangle$
 (b) $\langle 4 \diamond, K \spadesuit, 3 \clubsuit, 10 \heartsuit \rangle$.

How Big Can the Deck Be?

The trick works beautifully with a standard 52-card deck, but a natural question lingers: how large could the deck be? Could Badarika and Aalaya perform the same feat with 60 cards? A hundred?

Let us count. Suppose the deck has n cards. When the audience draws five and hands them to Badarika, the number of possible five-card hands is $\binom{n}{5}$. Badarika's message is an ordered row of four cards, and the number of such rows is $n(n-1)(n-2)(n-3)$. For the trick to be possible at all, there must be at least as many messages as hands: if two different hands forced Badarika to send the same message, Aalaya would see that message and have no way of telling them apart.

The inequality resolves to $n \leq 124$. That is a striking answer. The hard ceiling from information theory is 124 cards. A standard deck of 52 sits comfortably below it; in principle the

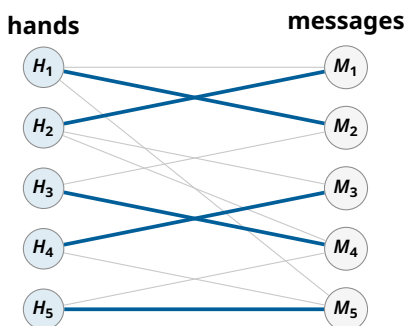
trick could work for a deck more than twice as large. Most people who meet the trick for the first time assume the numbers must be far tighter than this.

We should be careful, though. The condition $n \leq 124$ is necessary but not obviously sufficient. Having enough messages to go around does not automatically mean a valid strategy exists, because Badarika can only use the five cards she was dealt. She cannot lay down a message built from a card she never received. *Every hand must be matched to a message whose four cards all belong to that hand.* Showing that such a matching always exists takes a little more work.

The right framework is a bipartite graph (Figure 3). Place all possible five-card hands on one side, and all possible ordered four-card messages on the other. Draw an edge between a hand and a message whenever the four cards of the message all appear in that hand. A workable strategy for Badarika and Aalaya is exactly a *perfect matching* in this graph: each hand paired with a distinct message it contains. Does such a matching exist? It turns out that an important idea in graph theory which is known as the Hall's Marriage Theorem gives us a criterion: a perfect matching exists if and only if every set of k hands collectively reaches at least k messages.

To see if we meet this condition, checking each subset by hand would be hopeless, but fortunately the graph has a regularity that makes the condition automatic. Every five-card hand contains exactly five four-card subsets, and each of those can be arranged in 24 orders, so every hand-vertex has degree $5 \times 24 = 120$. On the other side, every ordered four-card message can be extended to a hand by appending any one of the remaining 120 cards, so every message-vertex also has degree 120. In this 120-regular bipartite graph, any collection of k hands sends out $120k$ edges in total; each message can absorb at most 120 of them, so those hands must





An edge means “the four cards of M_j all lie inside hand H_i .”

A **perfect matching** (thick edges) is a complete strategy.

At $n = 124$ every vertex has degree exactly 120, which forces such a matching to exist.

Figure 3: A toy picture of the real bipartite graph. At $n = 124$ the two sides are equal in size: $\binom{124}{5} = 124 \cdot 123 \cdot 122 \cdot 121 = 225,150,024$ vertices each, and the graph is 120-regular.

touch at least k distinct messages. Hall's condition holds for every subset at once, and a perfect matching exists.

The regularity carries a bonus. A finite d -regular bipartite graph splits into exactly d edge-disjoint perfect matchings—this is König's edge-colouring theorem, equivalently the integer form of the Birkhoff-von Neumann theorem (a nonnegative-integer matrix whose every row and column sums to d is a sum of d permutation matrices). Our graph is 120-regular, so it yields 120 edge-disjoint perfect matchings, and Badarika and Aalaya do not have just one valid strategy—they have at least 120 to choose from.

The catch is that the proof only confirms existence. At $n = 124$, the bipartite graph has 225,150,024 vertices on each side. The argument tells us a strategy exists; it gives no recipe a person could actually memorise and use. We give two methods that are performable: first a relaxed *warm-up* that reaches 124 by extending the ideas we had with the standard-deck and introducing one small physical gimmick, and then a more elegant scheme that works with 124 using nothing but the order of four cards, so it can be, in principle, performed over a phone call!

A Warm-Up: Four Suits and a Little Cheating

Begin by re-suiting the deck. Number the 124 cards 0 through 123. Give each card c

a “suit” $s = c \bmod 4$ —four suits, just as in a real deck—and a “rank” $r = \lfloor c/4 \rfloor$, the whole-number part of $c/4$, which runs from 0 to 30. Then $c = 4r + s$, and each suit holds exactly $124/4 = 31$ cards. The architecture is identical to a standard deck; only the number of ranks per suit has changed, from 13 to 31 (Figure 4).

The pigeonhole argument needs no adjustment. Five cards arrive from four suits, so at least two share a suit. Badarika picks any such pair: one will be hidden, the other revealed first, announcing the suit to Aalaya—exactly as before.

Now the clock. The 31 ranks sit on a clock face with 31 marks (Figure 5). Because 31 is odd, no two distinct marks sit exactly opposite each other, so any two different ranks have a well-defined shorter arc between them, always 1 to 15 steps. Badarika reveals the card she counts from, call it Y , and hides the card X that is k steps clockwise ahead, for some k between 1 and 15. The card Y announces the suit; the remaining job is to transmit k .

The three leftover cards carry this job part-way. In the agreed deck ordering they fall into Low, Medium, and High, and their six arrangements code a number from 1 to 6: LMH means 1, LHM 2, MLH 3, MHL 4, HLM 5, HML 6. That was the complete mechanism from the basic trick. The trouble here is that k can reach 15, and six arrangements reach only 6.



r	$\equiv 0$	$\equiv 1$	$\equiv 2$	$\equiv 3$
0	0	1	2	3
1	4	5	6	7
2	8	9	10	11
3	12	13	14	15
4	16	17	18	19
5	20	21	22	23
6	24	25	26	27
7	28	29	30	31
8	32	33	34	35
9	36	37	38	39
10	40	41	42	43
11	44	45	46	47
12	48	49	50	51
13	52	53	54	55
14	56	57	58	59
15	60	61	62	63
16	64	65	66	67
17	68	69	70	71
18	72	73	74	75
19	76	77	78	79
20	80	81	82	83
21	84	85	86	87
22	88	89	90	91
23	92	93	94	95
24	96	97	98	99
25	100	101	102	103
26	104	105	106	107
27	108	109	110	111
28	112	113	114	115
29	116	117	118	119
30	120	121	122	123

Figure 4: All 124 numbers re-suited as 4×31 . Each row is one rank r (the four consecutive numbers $\{4r, \dots, 4r + 3\}$); each column is one suit $s = c \bmod 4$, shaded from white ($\equiv 0$) to blue ($\equiv 3$). Two cards share a suit exactly when they lie in the same column.

So here we are going to cheat a bit. Lay the three cards along a straight line: say the near edge of the table. Now add a second signal --- either leave all three flush against the baseline, or nudge exactly one of them slightly forward (Figure 6). So we have four physical states in all—no nudge, or the first, second, or third card nudged—carry

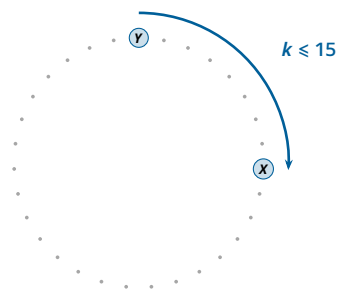


Figure 5: The 31 ranks of one suit, on a clock. Reveal Y and hide the card X that is k short steps ahead; since 31 is odd, k is always one of $1, \dots, 15$.

addends of 0, 6, 7, and 12. Aalaya reads the arrangement for a number from 1 to 6, reads the nudge for an addend, and adds them.

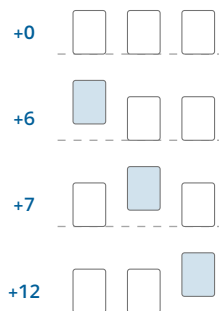


Figure 6: The nudge channel. All three cards resting on the baseline means add 0; lifting the first, second, or third card slightly means add 6, 7, or 12.

The arrangement picks a row of the table below and the nudge picks a column; the two add to give k .

	+0	+6	+7	+12
LMH = 1	1	7	8	13
LHM = 2	2	8	9	14
MLH = 3	3	9	10	15
MHL = 4	4	10	11	16
HLM = 5	5	11	12	17
HML = 6	6	12	13	18

The four addends do exactly what you want: 0, 6, 7, and 12, one per nudge option. Adding 0 covers arrangement values 1 through 6; adding 6 covers 7 through 12; adding 7 covers



8 through 13; adding 12 covers 13 through 18. Together these ranges blanket every k from 1 to 15. Values 8 through 13 are each reachable two ways, a harmless overlap, and the few totals above 15 (greyed in the table) are quiet ghosts, since k never reaches them.

If the choice of 12 seems a bit abrupt, you could also instead use the addends 0, 6, 7, 8. This will top out at 14, but you can still handle $k = 15$ with a flourish: lay the suit card face up and turn the other three face down, with an agreement beforehand that this special arrangement conveys a gap of 15. Make some banter about how you don't even need all the four cards, and your magic senses feel especially powerful today.

Let's work out a couple of examples. Suppose the five cards are 7, 19, 50, 88, 103. Their remainders mod 4 are 3, 3, 2, 0, 3, so Badarika works in suit 3 with the pair 7 and 19. Their ranks are 1 and 4 (since $7 = 4 \cdot 1 + 3$ and $19 = 4 \cdot 4 + 3$), so $k = 3$ —comfortably within the honest range. She reveals 7 and places the leftover cards 50, 88, 103 in MLH order, that is 88, 50, 103, all flush to the baseline. Aalaya sees 7 followed by 88, 50, 103, reads suit 3, rank 1, no nudge, MLH = 3, so $k = 3$, and the hidden card sits at rank $1 + 3 = 4$ in suit 3: that is $4 \cdot 4 + 3 = 19$.

Now let's try a scenario involving a nudge. Take the hand 2, 34, 11, 60, 99. Cards 2 and 34 share suit 2; their ranks are 0 and 8, giving $k = 8$. The arrangement alone reaches only 6, so Badarika combines: LHM encodes 2, and nudging the first leftover card adds 6, giving $2 + 6 = 8$. The leftover cards 11, 60, 99 in LHM order are 11, 99, 60, with the 11 nudged forward. Aalaya reads suit 2, rank 0, a first-card nudge (+6), LHM = 2, and concludes $k = 8$: the hidden card is rank 8 in suit 2, which is $4 \cdot 8 + 2 = 34$.

Note that performing this does not even need a deck of cards! Five numbers from 0 to 123 are called out, and Badarika jots them in a notebook. She writes four of them,

in order, on a board or a sheet of paper that everyone can see, and writes the hidden one on a small folded slip set aside as a *prediction*. physical nudge becomes a faint written mark instead—an underline, a dot, a casual scrawl—under exactly one of the three arrangement numbers (never the suit-giver), or under none. Aalaya walks in, reads the four numbers and the single mark (or notes the absence of it), names the hidden value, and the slip is opened to confirm.

If you work with the scheme of adding 6, 7, 8 then to communicate a 15, you can write the four numbers in any order, then make a cheesy remark along the lines of the prediction sharing a "very special connection" with one of these numbers, followed by erasing the three irrelevant numbers, and again you are done.

Problem 2

Suppose you are the assistant and the five numbers handed to you are 10, 2, 99, 42, 115. What will you hide? What will you write?

Problem 3

Suppose you are the magician and you find that the four numbers written in a row are 46, 17, 5, 88, and the 5 is underlined. You are the magician: name the hidden number.

Now: can we avoid the cheating? Imagine you had to perform this over a phone call to the magician, who's on speaker. Then you only have the order to work with, since using tonality of voice to communicate nudges is too unreliable, and coughing is suspicious. In a follow up article, we'll see how to work this magic with no cheating at all. Stay tuned!

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