

# The Fitch Trick: Solutions & Variations

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ସାରାଂଶ. TBD.

In the trick we presented in the last issue, Badarika the assistant studied five cards an audience handed her, slipped one away, and laid the other four face-up in a row. The magician Aalaya—who had been out of the room throughout—walked in, glanced at the four cards, and named the fifth.

These five cards could be cards from a standard deck of 52 cards, or cards from a deck of 124 cards. Last time, we saw how to work with 124 cards if we were allowed a physical gimmick. This time we'll see how to pull off the trick fair and square: so it's doable even over a phone call, where the assistant has to rely purely on card order and no extra signals.

## A Recipe for 124 Cards

So here is our upgrade to the pro version of this trick. Let's discuss the encoding first. When Badarika receives five cards, she sorts them mentally into increasing order and assigns them positions 0, 1, 2, 3, 4 from smallest to largest. She adds all five card-numbers together, divides by 5, and notes the remainder; call it  $i$ . She hides the card in position  $i$  and lays the other four face-up in a carefully chosen order. That is the complete hiding rule.

The reason this choice is clever comes down

to a small bookkeeping trick. Write the sorted hand as

$$c_0 < c_1 < c_2 < c_3 < c_4,$$

and suppose Badarika hides  $c_i$ . Aalaya sees the other four cards and calls their sum  $s$ . Now imagine setting those four visible cards aside and renumbering the remaining 120 cards in increasing order, from 0 up to 119.

What is the new number of the hidden card? Deleting cards only affects a card's label by removing the deleted cards below it. Since  $c_i$  was the  $i$ -th smallest card in the original five-card hand, exactly  $i$  of the visible cards lie below it. So, in the renumbered deck, the hidden card has new label

$$c_i - i.$$

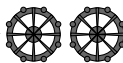
But  $i$  was chosen to be the total hand-sum modulo 5. Since that total is  $s + c_i$ , we have  $i \equiv s + c_i \pmod{5}$ , and therefore

$$c_i - i \equiv c_i - (s + c_i) \equiv -s \pmod{5}.$$

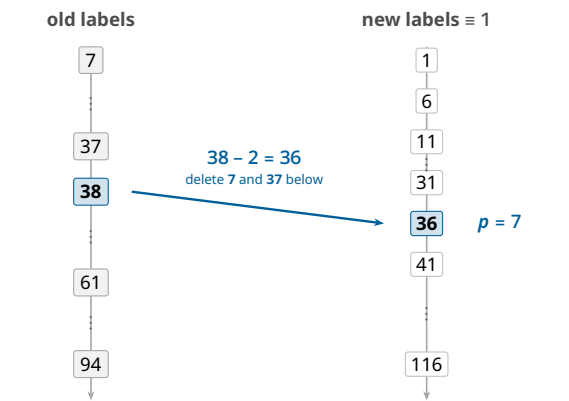
This is the key fact. The choice of which position to hide exactly compensates for the leftward shift caused by deleting the visible cards.

Potential to add a problem here.

Among 120 candidates, exactly  $120/5 = 24$  share any given remainder, so the forced remainder cuts Aalaya's search to 24 cards (Figure 1). The arrangement of the four revealed cards then says *which* of those 24 it is—and since four distinct cards admit exactly 24 orderings, the match is exact. Reading the arrangement as a number  $p$  from 0 to 23, Aalaya computes the hidden card's new number as  $5p + ((-s) \bmod 5)$ . Then she converts that new number to an actual card number by a simple counting-over step: she runs through the revealed cards in order, and each time a re-



vealed card sits at or below her running answer, she bumps the answer up by one.

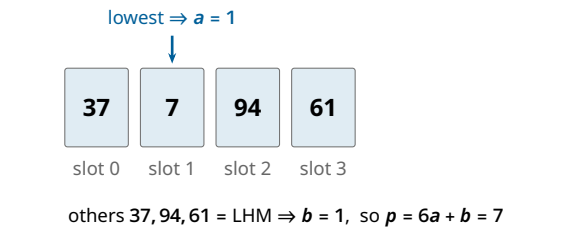


**Figure 1:** A concrete picture of the renumbering in the worked example. The shown cards below 38 are 7 and 37, so deleting the shown cards changes the hidden card's label from 38 to  $38 - 2 = 36$ . Since  $36 \equiv 1 \pmod{5}$ , it appears as the  $p = 7$  entry among the 24 possible new labels 1, 6, 11, ..., 116.

Need some better pictures here.

A concrete run-through helps. Suppose Badarika lays down four cards in this order: 37, 7, 94, 61. Their sum is  $s = 199$ . Since 199 leaves remainder 4 on division by 5, the forced remainder is  $(-4) \bmod 5 = 1$ . The arrangement of the four cards encodes the value  $p = 7$  (the mechanism comes next). So the hidden card's new number is  $5 \cdot 7 + 1 = 36$ . To recover the actual card, start from 36: the revealed card 7 is at most 36, so the answer becomes 37; the revealed card 37 is now at most 37, so the answer becomes 38; the cards 61 and 94 sit above 38, so no more bumps. **The hidden card is 38.**

Now for the only step that might seem fiddly: turning an arrangement of four cards into a number from 0 to 23, and back, without any factorial arithmetic. Write  $p = 6a + b$ , where  $a$  runs over 0, 1, 2, 3 and  $b$  over 0, 1, 2, 3, 4, 5; every  $p$  in 0, ..., 23 factors this way uniquely. Badarika places the *lowest* of her four cards



**Figure 2:** The encoding of the permutation: the slot of the lowest card gives  $a$ ; the LMH pattern of the other three gives  $b$ ; and  $p = 6a + b$ .

into slot number  $a$ , counting slots 0, 1, 2, 3 from left to right. She arranges the other three cards in the remaining slots using the very same Low-Medium-High table from the basic trick: the patterns LMH, LHM, MLH, MHL, HLM, HML encode  $b = 0, 1, 2, 3, 4, 5$ . To decode, Aalaya notes which slot holds the lowest card (giving  $a$ ), reads the three-card pattern in the remaining slots (giving  $b$ ), and computes  $p = 6a + b$ .

The worked example closes neatly on itself (Figure 2). The four revealed cards are 37, 7, 94, 61. The lowest, 7, sits in slot 1, so  $a = 1$ . The other three, read left to right, are 37, 94, 61—that is Low, High, Medium, the LHM pattern, so  $b = 1$ . Hence  $p = 6 \cdot 1 + 1 = 7$ , exactly the value the calculation required.

Add the factorial method for encoding a permutation as well.

One last observation. At 124 cards the arithmetic comes out perfectly: the number of five-card hands,  $\binom{124}{5}$ , equals the number of ordered four-card messages,  $124 \cdot 123 \cdot 122 \cdot 121$ . There is no slack at all: the correspondence between hands and messages is a bijection, and not one of the 24 orderings is spare.

### Problem 1

You are given the numbers: 10, 3, 16, 121, 42. What will you hide? What will you show?

## Problem 2

The four numbers laid in a row are 9, 40, 2, 88. Name the hidden number.

### Variations on the Theme

The scheme we built has two moving parts: Badarika's choice of which card to hide, and her arrangement of the four she shows. Both levers can be adjusted, and each adjustment opens a different room.

**The coin flip (Elwyn Berlekamp).** Stretch the deck to 64 cards. After Badarika lays out four cards from a hand of five, 60 cards remain unseen; the hiding rule leaves the hidden card in a residue class of exactly  $60/5 = 12$ . The four shown cards still arrange in 24 ways, and 24 is exactly twice 12. That spare factor of two carries one extra bit: Aalaya names the hidden card *and* announces the result of a coin an audience member flipped in secret before the show. The same four-card layout does double duty, and the audience sees nothing unusual.

## Problem 3

Explain the magic number 64. Show four cards of a deck of  $D$  and ask why the 24 orderings can carry both the hidden card *and* a coin bit only when  $D \leq 64$ .<sup>a</sup>

<sup>a</sup>After four cards are shown,  $D - 4$  remain, and the hiding rule pins the hidden card to a residue class of  $(D - 4)/5$  candidates. Naming it *and* a coin flip needs  $24 \geq 2 \cdot (D - 4)/5$ , that is  $D - 4 \leq 60$ , so  $D \leq 64$ —with equality exactly at 64, where  $(D - 4)/5 = 12$  and  $2 \cdot 12 = 24$ .

**Face up, face down (Colm Mulcahy's "Ups and Downs").** Keep the suit-and-pair logic, so again only a number from 1 to 6 must be sent. Instead of encoding it through the ordering of three cards, Badarika encodes it through their *orientation*: D for face-down, U for face-up. The six patterns DDU, DUD, DUU, UDD, UDU, UUD represent 1 through 6 (Figure 3), and two patterns are left over. Since

UUU never appears, Aalaya can treat it as a private signal to switch back to the original trick; the performer shifts modes mid-show with a casual "Shall we make it harder and only show some of the cards?" and nobody notices the mechanism changed. Since DDD never appears either, at least one card always lies face up, and that first face-up card can itself announce the suit. The dedicated suit card becomes redundant: only *three* cards need be shown at all, and the fourth can be set aside with a theatrical flourish.



**Figure 3:** "Ups and Downs." Three cards, each face-up (U) or face-down (D), send a number from 1 to 6. The patterns UUU and DDD are never used—each becomes a hidden bonus.

**Hiding the tell (Mulcahy's "Suit Alterations").** A careful watcher of the basic trick notices, after a few rounds, that the suit-giving card always sits first. Here is the promised fix. Once Badarika has her four cards, she adds their values and reduces modulo 4 (using 4 in place of 0); the result names *which position* carries the suit, and the other three carry the number as usual. For example, the displayed cards **4♠**, **Q♣**, **5♥**, **10♦** sum to  $4 + 12 + 5 + 10 = 31 \equiv 3 \pmod{4}$ , so the third card, the **5♥**, names the suit; the remaining three in HLM order communicate 6, and the hidden card is the **8♥**.

**When the volunteer chooses (after Gardner's "Eigen's Value").** Now the volunteer, not Badarika, keeps one card back, so Badarika's best lever is gone. She has only the ordering of four cards—24 arrangements—to point at one of 48 unknowns, and 24 can narrow 48 down to at most two. The resolution is stagecraft. With a fixed total order agreed in advance, the four displayed cards single



out two candidates (say the 15th and 39th in the residual ordered list of 48); Aalaya quietly brings one to the top of the deck and the other to the bottom. Once the chosen card is named aloud, she turns over either the top or the bottom card to reveal it. The deciding bit is smuggled in by *which end* she turns over, while every eye is on the volunteer.

**Pushing to the extreme (Tom Edgar).** With a “one-way” deck—backs with no rotational symmetry—each card can point in four directions rather than two. Two face-down cards of this kind, where one may also sit slightly atop the other, produce  $2 \times 4 \times 4 = 32$  distinct signals; combined with modular arithmetic on the single revealed card's value, even *two* cards suffice for the 52-card trick. There is even a one-card version using two differently-coloured decks, where the relative placement and orientation of a single card carry the whole message. These feel more like combinatorial puzzles than performances, but they show how far the core idea stretches.

**The general pattern.** Behind every variation sits the same inequality. Suppose the audience draws a hand of  $n$  cards, Aalaya is shown  $r$  of them, and she must name the rest. A strategy exists exactly when

$$\frac{(N-r)!}{(N-n)!} \geq n!,$$

where  $N$  is the deck size. When  $r = n - 1$  (all but one card shown) the largest workable deck satisfies  $N = n! + n - 1$ ; for  $n = 5$  this gives 124, and at that maximum every hand of five matches exactly one arrangement of four—no slack, no redundancy. Such a “perfect” trick is the card-magic cousin of a perfect code.

Several questions sit at the edges of this picture. For which pairs  $(n, r)$  does a perfect deck size exist—one where hands and messages match with no remainder, a clean Diophantine condition  $\frac{(N-r)!}{(N-n)!} = n!$ ? How many essentially different strategies are there for given  $(n, r, N)$ , once one strips away relabellings of

the deck? And whenever a strategy exists, is there always one simple enough to carry in one's head, or does existence sometimes outrun any hope of performance?

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