

The Fitch Trick and Variants

ଫିଚ୍ ଡାସ ଖେଳ ଏବଂ ଏହାର ରହସ୍ୟ

ନୀଳଧାରା ମିଶ୍ର • Neeldhara Misra

ସାରାଂଶ. ଯାଦୁକର ପ୍ରଥମେ ଦର୍ଶକଗଣଙ୍କୁ ଏକ ତାସ ବିଡ଼ାକୁ ଭଲଭାବେ ମିଶାଇ ସେଥିରୁ ଯେକୌଣସି ପାଞ୍ଚଟି ତାସ ପତ୍ର ବାଛିବା ପାଇଁ ନିର୍ଦ୍ଦେଶ ଦିଅନ୍ତି। ଏହାପରେ ଯାଦୁକର ସେହି ପାଞ୍ଚଟି ତାସଗୁଡ଼ିକୁ ନିରୀକ୍ଷଣ କରି ସେଥିରୁ ଗୋଟିଏ ତାସକୁ ଲୁଚାଇ ଦିଅନ୍ତି ଏବଂ ବାକି ଚାରୋଟି ତାସକୁ ଖୋଲାଭାବରେ (ମୁହଁ ଉପରକୁ କରି) ଗୋଟିଏ ଧାଡ଼ିରେ ସଜାଇ ରଖନ୍ତି। ତା'ପରେ ସେ ବାହାରେ ଅପେକ୍ଷା କରିଥିବା ନିଜର ଜଣେ ସହଯୋଗୀଙ୍କୁ ଡାକନ୍ତି। ସହଯୋଗୀ ଜଣକ ଆସି ସେହି ଚାରୋଟି ତାସଗୁଡ଼ିକ ଦେଖିବା ମାତ୍ରେ ହିଁ ଲୁଚାଯାଇଥିବା ତାସଟିକୁ ସଠିକ୍ ଭାବରେ ଚିହ୍ନଟ କରିଦିଅନ୍ତି। ଏହା କିପରି ସମ୍ଭବ?

Aalaya the magician and her assistant Badarika are about to perform an impossible feat of telepathy. To prove there is no foul play, Badarika leaves the room. She leaves behind her phone, smartwatch, and any other gadgets to ensure no one can communicate with her.

Then the audience picks out five cards from a shuffled deck and hands them to Aalaya. She studies the five cards, secretly removes one, and in a single fluid motion arranges the remaining four face-up neatly in a row.

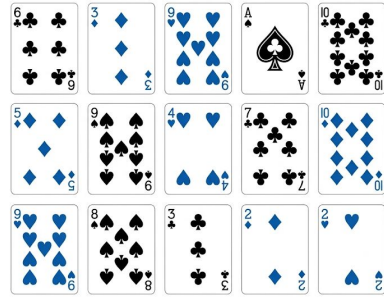
Once everything is set, Badarika is called back into the room. She carefully observes the four visible cards... and, to everyone's amazement, correctly identifies the fifth card—the one she has never seen. How is this possible?

The Mathematics Behind the Magic

Notice that Badarika has to guess one out of 48 possible cards, since the deck has 52 cards of which four are face up. On the other hand, Aalaya can arrange the four cards in $4! = 24$ different orders. This feels like we are falling short by a factor of two!

But Aalaya actually has a point of leverage that we have not yet accounted for: she can *choose* the card that she wants to set aside.

In other words, she is not just choosing an order for the four visible cards—she is also choosing which of the five cards to hide. This gives her $4! \times 5 = 120$ possible signals, more than enough to distinguish between the 48 possibilities Badarika needs to resolve.



In all these three cases, Aalaya shows the first four cards and Badarika guesses the fifth. What do you notice? Can you spot a pattern?

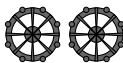
So how do we think about a signaling strategy? Let's break down the numbers:

- There are $\binom{52}{5} = 2,598,960$ possibilities for what the audience can hand Aalaya. Call this set **P**.
- The number of ordered sequences of four cards Aalaya can place on the table, say **Q**, is $52 \cdot 51 \cdot 50 \cdot 49 = 6,497,400$.

Each of the 120 possible signals corresponds to an element of **Q**.

What we want is a rule that assigns each $\alpha \in \mathbf{P}$ (what the audience does) to a unique and feasible choice of $\beta \in \mathbf{Q}$ (what Aalaya would do). Badarika then comes in, sees β , works backward to α , and identifies the missing card.

In fact, since **Q** has more elements than **P**, such a mapping definitely exists. But more interestingly, can you find a simple rule they could execute live, under pressure, with an audience watching?



Problem 1

We write cards by rank and suit, with $S, H, C,$ and D denoting Spades ($\spadesuit$), Hearts ($\heartsuit$), Clubs ($\clubsuit$), and Diamonds ($\diamondsuit$). Let $a = \langle 4S, 5H, 2C, 7C, 6D \rangle$ be a hand given to Aalaya. Which β does a map to under your rule?

Pushing the Limits

You may have noticed that there is a considerable gap between the number of signals (6,497,400) and the number of possibilities (2,598,960) we calculated earlier. This raises a natural question: can we turn this extra room into a more impressive trick? The effect remains the same: four cards are shown in a fixed order, and a fifth is guessed. But could the audience be choosing from a *larger* deck?

If the audience is choosing from a deck of size n in general, then there are $\binom{n}{5}$ possible sets of cards they could hand to Aalaya, while the number of ordered sequences of four cards she can display is $n \cdot (n-1) \cdot (n-2) \cdot (n-3)$. For the trick to work, the space of signals must be at least as large as the space of possibilities—otherwise, two different hands would have the same signal introducing ambiguity.

Problem 2

From the inequality

$$n(n-1)(n-2)(n-3) \geq \binom{n}{5},$$

conclude that $n \leq 124$.

This shows that $n \leq 124$ is a *necessary* condition for the trick to work, but not necessarily a sufficient one. The next challenge is to show that the trick can, in fact, be made to work with a deck this large.

So once you have figured out how to do it with a standard deck, try pushing further. Can you extend your method to larger decks, perhaps all the way up to 124?

Some Variations

A variation of this trick, performed by **Elwyn Berlekamp**, uses a deck of size 64 and adds an extra twist: Badarika also correctly guesses the outcome of a coin flip made by an audience member. If you can figure out the 124-card version, you should start to see how to handle the version with the coin as well.

You might also wonder whether the reveal really has to rely only on the *ordering* of the four cards. A variation suggested by **Colm Mulcahy** expands the signaling method: the cards are not only ordered but some are turned face up while others are face down. This allows the hidden card to be encoded using only *three* cards instead of four! This makes for great theatrical flair, as discarding a card becomes part of the performance.

We can frame the trick more generally as follows: the audience selects p cards from a deck of size n , and the magician reveals q of them. For which values of n , p , and q is such a trick possible? And when it is possible, how many distinct signaling strategies are there?

Please feel free to email your ideas to the author at the email below. And keep an eye out for future issues which will bring solutions and new puzzles!



Left: link to a Youtube demonstration of the trick in its basic form. Right: link to an interactive where you can practice additional examples.

 Associate Professor, CSE; IIT Gandhinagar

 neeldhara.misra@gmail.com

