

A Trick Fitch for Two

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Everyone loves a good mathematical card trick, and one of the most famous is the Fitch Cheney five-card trick. For this trick, a magician and an assistant secretly encode information in a sequence of five randomly selected cards that allows one of them to guess one card. The trick can even work if the deck contains 124 cards.

In the February 2003 issue of *Math Horizons*, Colm Mulcahy described the full details of the classic Fitch Cheney trick and produced a four-card version. A decade later in the November 2014 issue, Mulcahy used a “one-way deck” (one without a rotationally symmetric back) to reduce the number of cards to three. It’s been another decade, so I thought I’d follow up and use various numeration systems, coupled with modular arithmetic, to reduce the number of cards to two!

To perform this trick, a magician sends an assistant out of the room and asks an audience member to randomly select two cards from a standard 52-card deck. Let’s say the spectator chooses the three of diamonds and the queen of clubs. The magician glances at the two cards and lays them down, side-by-side, on the table as pictured on the left in figure 1. The magician’s assistant returns to the room and asks the spectator to reveal one of the two cards. The spectator turns over the three of diamonds, and, after a moment, the assistant claims that the second face-down card is the queen of clubs—the magician has used telepathy to transmit the result!

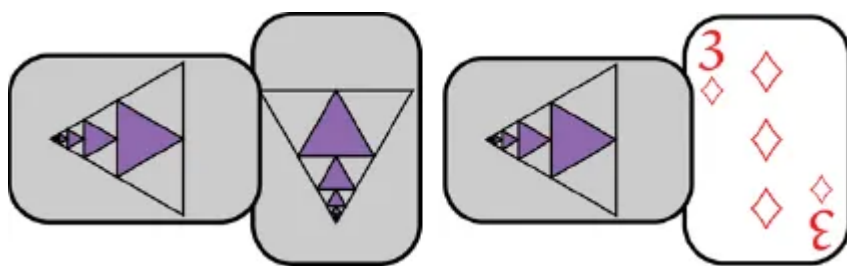


Figure 1. One configuration of face-down cards for the trick, before an... »

Just Two Cards Works?

It is not surprising that this trick, like the three-card variant, requires a one-way deck. However, we also make use of the fact that a one-way deck can be distinctly aligned in *four*

ways, with the “top” pointing either north, east, west, or south. Here, we use the term “top” to signify some agreed-on side of the asymmetric design. Using two arranged cards allows for a total of $4 \cdot 4 = 16$ configurations of face-down, side-by-side cards. Still, 16 pieces of information are not nearly enough to encode all possible 1,326 distinct pairs of cards.

The second gimmick relies on the relative placement of the two cards: We either place the left card on top of or underneath the right card (if ever so slightly). In figure 1 the left card barely overlaps the right card so that the left card is on top. With this technique, we obtain a total of $2 \cdot 4 \cdot 4 = 32$ different configurations of the two cards. We encode each of these configurations by a triple (T, C_1, C_2) , where T is either L or R

(depending on if the left or right card is on top) and both C_1 and C_2 take the values N , E , S , or W depending on which direction the “top” of the corresponding card points (C_1 for the left card and C_2 for the right card). For example, we encode the configuration in figure 1 as (L, W, S) .

Code	L	R	N	E	S	W
Value	0	1	0	1	2	3

Table 1. Mapping card configurations to numbers. »

Using the coding in table 1, each configuration of cards corresponds to a triple (a_1, a_2, a_3) where $0 \leq a_1 \leq 1$ and $0 \leq a_2, a_3 \leq 3$. Given a triple

of this specified form, we obtain an integer between 0 and 31 by determining $n = a_1 \cdot 1 + a_2 \cdot 2 + a_3 \cdot 8$.

The configuration of cards in figure 1 yields the triple $(0, 3, 2)$ and the integer $0 \cdot 1 + 3 \cdot 2 + 2 \cdot 8 = 22$.

Moreover, given an integer m between 0 and 31, we recover the appropriate triple (a_1, a_2, a_3) by two applications of the division algorithm:

$$m = 2 \cdot q_1 + a_1$$

$$q_1 = 4 \cdot a_3 + a_2.$$

We note that this representation of the integers between 0 and 31 is a *mixed-radix positional number system* with positions 1, 2, and 8.

Finishing the Trick

Having only 32 card configurations still seems insufficient compared to the number of pairs of cards. Remember, though, that the assistant also asks for one card to be revealed. We use this card to unlock the value of the other.

Each card in a standard deck can be interpreted as an ordered pair (V, S) , where V and S are the numerical values of the card and suit (see table 2). Therefore, each card is uniquely represented by an integer $C = S \cdot 13 + V$ between 0 and 51. Also, given an integer C between 0 and 51, we recover the corresponding pair by performing the division algorithm with C and 13.

This is again another positional number system, namely the base-13 positional system. Revisiting our original example, the three of diamonds corresponds to the tuple $(3, 3)$ and thus the integer $3 \cdot 13 + 3 = 42$, while the queen of clubs corresponds to the tuple $(12, 0)$, which results in the integer $0 \cdot 13 + 12 = 12$.

When we arrange the integers 0 to 51 in a circular fashion (figure 2), they represent a number system known as the integers modulo 52. We add two integers modulo 52 by

Suit	Clubs	Hearts	Spades	Diamonds
Value	0	1	2	3

Card	King	Ace	Jack	Queen
Value	0	1	11	12

Table 2. Numerical values for suits and nonnumbered cards. »

adding them as usual and reducing to the remainder obtained on dividing the result by 52. This is equivalent to just moving around the circle counterclockwise ($50 + 3 = 1$).

By thinking of a deck of cards this way, we see that any pair of cards is at most

26 units apart. In our example, the numbers 12 and 42 are 30 units apart if we travel from 12 to 42 counterclockwise, but they are $(52 - 42) 12 = 10 + 12 = 22$ units apart if we travel from 12 to 42 clockwise. Thus, given two distinct cards, we can assume the left card has value a and the right card has value b where $d = a + (52 - b) \bmod 52$ satisfies $0 < d \leq 26$. If the cards were the two of hearts (15) and the nine of clubs (9), then the difference $15 - 9$ is less than 26 so the left card would be the two of hearts. If the cards were the two of diamonds (41) and the nine of clubs (9), the left card would be the nine of clubs because $41 - 9$ is greater than 26. Moreover, if we know d and either a or b , we can use the equation to find the other value (either a or b).

We now have the tools to completely describe our trick. Using the spectator's two cards with values a and b , the magician computes $d = a + (52 - b) \bmod 52$ (recall that we order a and b so that $0 < d \leq 26$). Then, the magician computes the triple (a_1, a_2, a_3) with $d = a_1 \cdot 1 + a_2 \cdot 2 + a_3 \cdot 8$ and uses table 1 to obtain the triple (T, C_1, C_2) providing the instructions for how to position the two cards on the table (a and b correspond to left and right cards, respectively).

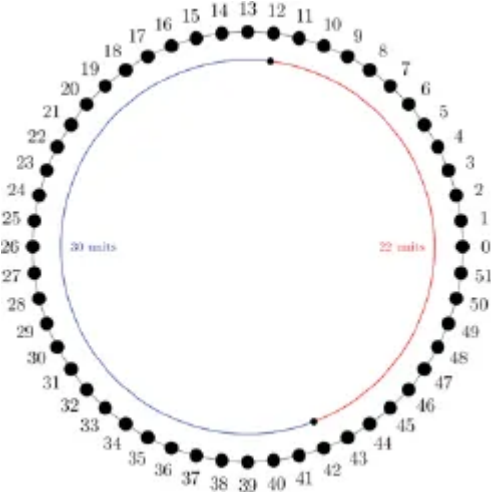


Figure 2. The integers modulo 52. Any ... »

The assistant returns, reads off the configuration (T, C_1, C_2) , and uses table 1 to compute both (a_1, a_2, a_3) and $d = a_1 \cdot 1 + a_2 \cdot 2 + a_3 \cdot 8$. When the spectator turns over their chosen card, the assistant knows whether it is a or b (left or right) and then uses the equation $a + (52 - b) = d \bmod 52$ to determine the unknown integer (suppose it's a). At this point, the assistant uses the division algorithm to divide a by 13 resulting in $a = S \cdot 13 + V$ which, via table 2, allows the assistant to identify the card corresponding to (V, S) .

Amazingly, we have used arithmetic modulo 52 along with two different positional representations of integers to perform this two-card Fitch Cheney variant.

Give it a try yourself. Suppose you are the assistant, and you are given the arrangement on the left of figure 3. When you ask the spectator to turn over a card, they reveal that the left card is the six of hearts. What is the other card?

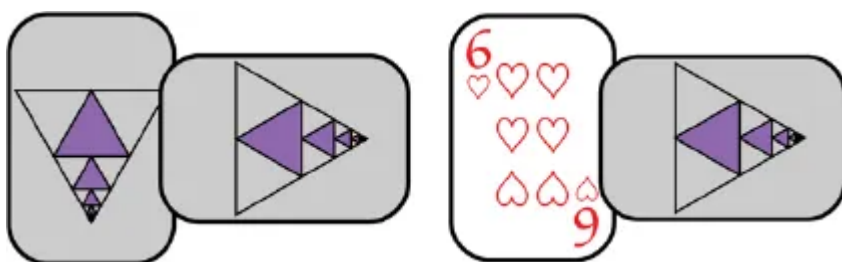


Figure 3. What is the card on the right? »

Push to a Single Card

A natural question arises: Can we perform a similar trick with only one card? Here is a possible technique, where we use two full decks of cards, each with backs that are not

rotationally symmetric, with different colors.

After the assistant leaves the room, the spectator chooses a random card from the red deck, and the magician finds the exact same card in the blue deck. The spectator keeps their card, and the magician lays the blue card face down. For this trick, the magician will (secretly) use the placement of the two decks of cards. With the two decks of 51 cards (different colors) and the single card, which we will think of as a third deck with one card, we have a total of $3! \cdot 6$ permutations of decks; all six are shown in figure 4. Note that the magician can be clever about where to place the decks so that they don't appear to be relevant to the trick; all that matters is their relative position to each other and the single card.

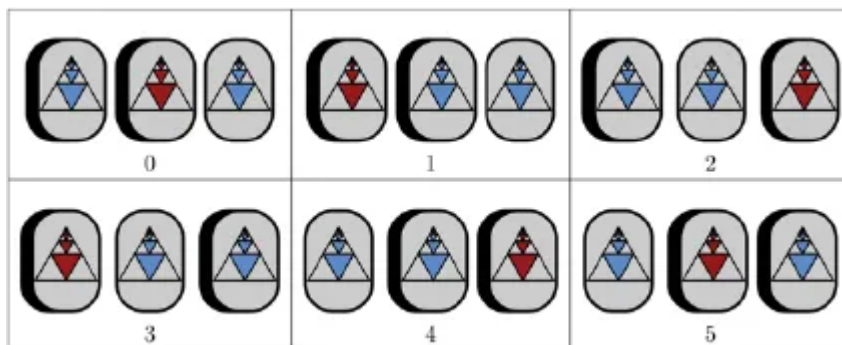


Figure 4. The six permutations of the three decks (one with only one card... »

Each deck can also be oriented with its "top" facing north or facing south. There are 48 such configurations, and the magician can again use a positional number system to encode, uniquely, each configuration by one of the integers between 0 and 47 as follows.

The permutations listed in figure 4 are numbered 0 to 5, and each deck (including the one-card deck) gets a value of 0 or 1 depending on if its top is pointing north or south, respectively. Thus, each deck configuration can be represented by a tuple of the form (c, d_1, d_2, p) where $0 \leq p \leq 5$ represents the permutation and d_1, d_2 , and c represent the orientation of blue deck, the red deck, and the single card, respectively, with $0 \leq d_1, d_2, c \leq 1$. In turn, each such tuple determines an integer by

$$n = c \cdot 1 + d_1 \cdot 2 + d_2 \cdot 4 + p \cdot 8.$$

Conversely, given an integer m , we can reconstruct the tuple by using a greedy algorithm: Find the largest integer of the form $8 \cdot p$ less than or equal to m and compute $q_1 = m - 8 \cdot p$. Then, find the largest integer of the form $4 \cdot d_2$ less than or equal to q_1 and compute $q_2 = q_1 - 4 \cdot d_2$. Finally, use the division algorithm to obtain $q_2 = 2 \cdot d_1 + c$ so that $m = c \cdot 1 + d_1 \cdot 2 + d_2 \cdot 4 + p \cdot 8$.

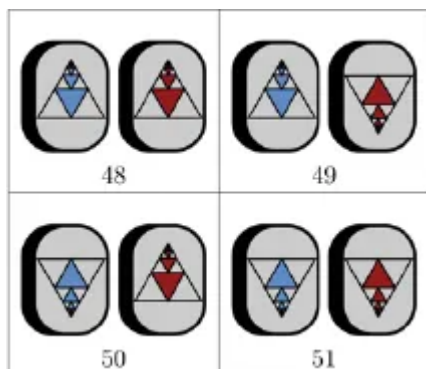


Figure 5. Two deck configuration... »

This correspondence, along with the method of converting cards to integers between 0 and 51 from the previous section, allows for almost full communication between the magician and the assistant. To deal with the fact that 48 is just short of 52, we note that if the chosen card is one of the cards corresponding to 48–51 by table 2, the magician simply leaves the chosen card on top of the blue deck and only places the two decks on the table. Each deck still can still be oriented to point north or south. The four configurations are listed in figure 5 with the appropriate card number.

Here's a final challenge: Determine the selected card if the magician laid out the cards in figure 6. The answers to the magical challenges are hidden elsewhere in this issue. Happy hunting!

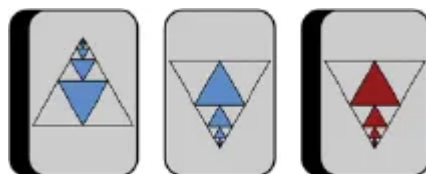


Figure 6. A configuration of three decks. What is the spectator's card? »

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